Quasicircular Ground Tracks for Geosynchronous Earth Satellites

Jerry K. Cline*
McDonnell Douglas Astronautics Company, St. Louis, Missouri
and
Philip C. Almes†
Wayland Baptist University, Plainview, Texas

A set of sufficient conditions on the Keplerian elements of a geosynchronous (24 h) Earth satellite to produce a near-circular ground track on the rotating planet is derived. Once the required semimajor axis of the orbit and argument of perigee are found, a simple relationship between the eccentricity and the inclination of the orbital plane to the equatorial plane is obtained. The equation of the ground track is obtained. An analysis of the deviations of the ground tracks from circles is made using the root mean square error, the mean absolute error, and the area between the ground track and a circle. It was found that the first two tend to zero faster than does the quasiradius of the ground track (the inclination), while the area tends to zero faster than the square of quasiradius. Since the number of positions for satellites in geosynchronous orbit is limited, placing them in orbits that produce near-circular ground tracks centered over the equator can greatly increase the capacity.

Nomenclature

a = semimajor axis
$A(i)$ = area between ground track and circle
e = eccentricity
E = eccentric anomaly
f = true anomaly
i = inclination of the orbital plane with respect to the equatorial plane
n = angular spin rate of the Earth on its axis, rad/s
P = period
r = radial distance from the satellite to the center of the Earth
t = time from perigee, s
T = time of perigee passage
$\Delta(\phi)$ = difference in longitude of ground track and circle for a given latitude
θ = inertial longitude
θ_r = Earth (noninertial) longitude
θ_G = inertial longitude of the Greenwich meridian at $t=0$
θ_0 = inertial longitude at the center of the circle
θ' = longitude of circle for a given latitude
μ = gravitational constant for the Earth
ϕ = latitude from the equator
ω = argument of perigee
Ω = longitude of ascending node

Introduction

To a first approximation, the orbital path of a satellite around a spherical Earth is determined by its Keplerian elements a, e, i, ω, Ω, and T. The size and shape of the orbit are fixed by a and e, while its orientation is specified by i, ω, and Ω. The actual location of the satellite in the orbit is determined by the time t since perigee passage T. The intersection of the geocentric radius vector with the Earth's surface as the satellite moves in its orbit is called the orbital ground track. Due to the interplay between the motion of the satellite in its orbit and the Earth's rotation on its polar axis, the shapes of orbital ground tracks can be quite varied. Low-altitude circular orbits with nonzero inclination produce the sinusoidal ground tracks on Mercator map projections, which became familiar during the early manned space programs. Closed curves can be obtained by choosing orbits with 24 h periods (called synchronous orbits). These can be simple closed curves, complicated lemniscate figures, or even a single point in the case of a synchronous, circular, equatorial orbit. The purpose of this paper is to develop sufficient conditions for the Keplerian elements to produce a particular simple closed curve as a ground track, namely a near circle.

Designing an orbit so as to produce a certain type of ground track is well known. For example, Takahashi[1] showed how to create tracks so that the satellite remains nearly stationary for several hours over points in the northern hemisphere. In 1967, Betz[2] reported an analytical development for a ground track that included the track for a circular synchronous orbit. Leonard and Woodford[3] have suggested that a set of four geosynchronous satellites with nearly circular ground tracks be used as the basis of a navigation system. This design (called the "inverted Y deployment") has been studied by Bierman,[4] who was mainly concerned with tracking accuracies. Nearly circular ground tracks were obtained for $i=60$ deg and $e=0.6$ for a perigee angle of 90 deg by Bielkowicz.[5]

Sufficient Conditions for a Quasicircular Ground Track

The overall approach will be to force the ground track to match the circle at equatorial points and points of latitude extrema. We will then determine the degree of error produced by such a configuration.

Let (x, y, z) be an inertial geocentric coordinate system, with the x axis pointing toward the vernal equinox, the z axis in the direction of the Earth's north pole, and the y axis completing the right-handed system. The orientation of a satellite orbit in this coordinate system is shown in Fig. 1. The angle θ is the inertial longitude measured eastward from the x axis and ϕ is latitude from the equator (north positive).

The angle Ω positions the ground track relative to the x axis and T is a time reference. In other words, the shape of the ground track is independent of Ω and T, so without any loss of generality it will be assumed that $\Omega=0$ and $T=0$. The

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*Technical Specialist, Engineering Division.
†Associate Professor of Mathematics.

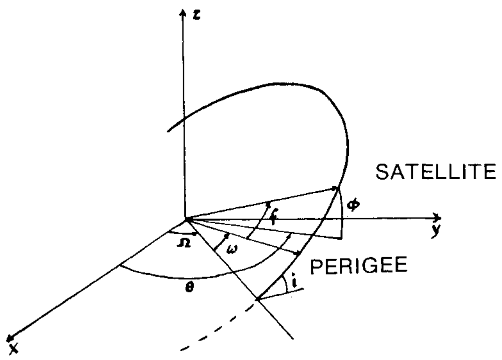

Fig. 1 Orbit geometry.

characteristics of the ground track curve then depend only on the four remaining elements a, e, i, and ω.

Conditions on a and ω are easily found. In order that the ground track curve be closed, it is sufficient that the orbit be synchronous. The period P is related to the semimajor axis a by the familiar equation

$$P = 2\pi\sqrt{a^3/\mu} \tag{1}$$

where $\mu = 3.986 \times 10^5$ km^3/s^2 is the gravitational constant for the Earth. Hence, for a 24 h period

$$a = 42{,}164 \text{ km} \tag{2}$$

and so the semimajor axis is assumed fixed at this value. To ensure that the ground track is symmetrical with respect to a longitudinal line, it is sufficient that

$$\omega = \pm\pi/2 \tag{3}$$

See Fig. 2, which shows the "pear-shaped" ground track. If $\omega = -\pi/2$, the ground track is reflected about the equator with the bulge on the perigee side and the contraction on the apogee side. It now remains to find conditions on e and i to give a near-circular ground track. From spherical trigonometry come the following relations:

$$\tan\phi = \tan i \sin(\theta - \Omega) \tag{4}$$

$$\cos(\omega + f) = \cos\phi\cos(\theta - \Omega) \tag{5}$$

$$\sin\phi = \sin(\omega + f)\sin i \tag{6}$$

where f is true anomaly (see Fig. 1). With $\omega = \pi/2$ and $\Omega = 0$, Eqs. (4-6) become

$$\tan\phi = \tan i \sin\theta \tag{7}$$

$$\sin f = -\cos\phi\cos\theta \tag{8}$$

$$\sin\phi = \cos f \sin i \tag{9}$$

Now let θ_r be the Earth (noninertial) longitude of the subsatellite point relative to the rotating Earth. Then, θ_r and θ are related by

$$\theta_r = \theta - (nt + \theta_G) \tag{10}$$

where t is time from perigee in seconds, $n = 2\pi/86164$ the angular spin rate of the Earth on its axis in radians per second, and θ_G the inertial longitude of the Greenwich meridian at $t = 0$. Note that $\dot{\theta} = n$ (mean motion) for a synchronous satellite. The latitude ϕ of the ground track will, by necessity, reach both $+i$ and $-i$ as extrema; hence, any circular ground track must have its center on the equator. Let θ_0 be the inertial longitude of the center of the circle and the inclination i be fixed. From spherical trigonometry, the equa-

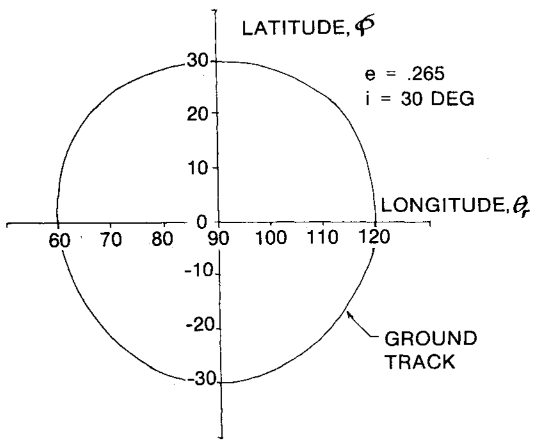

Fig. 2 Sample circular ground track.

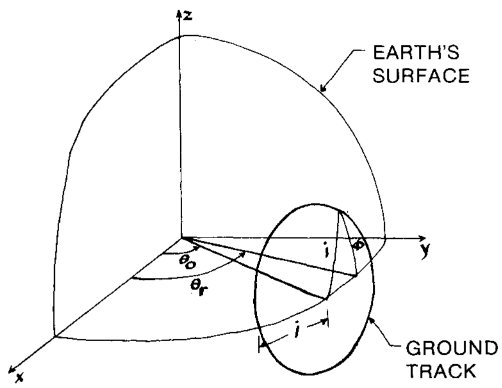

Fig. 3 Circular ground track geometry.

tion of the circle centered at $\theta_r = \theta_0$, $\phi = 0$ with constant angular radius i (see Fig. 3) is

$$\cos\phi\cos(\theta_r - \theta_0) = \cos i \tag{11}$$

Assume the satellite is at perigee so that $t = 0$ and $f = 0$. Then Eq. (9) gives $\phi = i$ and Eq. (7) yields $\theta = \pi/2$, while Eq. (10) gives $\theta_r = \theta - \theta_G = \pi/2 - \theta_G$. Finally, Eq. (11) yields $\theta_r = \theta_0$. Hence,

$$\theta_0 = \pi/2 - \theta_G \tag{12}$$

and using Eqs. (10) and (12), Eq. (11) becomes

$$\cos\phi\sin(\theta - nt) = \cos i \tag{13}$$

Now Eq. (13) must hold for all t and, in particular, it must hold for $t = t^*$ when the true anomaly $f = \pi/2$. From Eqs. (7-9), when $f = \pi/2$, $\phi = 0$, and $\theta = \pi$. Hence, at $t = t^*$, Eq. (13) becomes

$$\sin(\pi - nt^*) = \cos i$$

or

$$\pi/2 - nt^* = i \tag{14}$$

Now Kepler's equation for the orbit is

$$nt = E - e\sin E \tag{15}$$

where E is the eccentric anomaly. The radial distance r of the satellite from the center of the Earth is a function of the true anomaly f, given by the conic equation

$$r = a(1 - e^2)(1 + e\cos f)^{-1} \quad (16)$$

and r, f, and E are related by

$$r = a(1 - e\cos E) \quad (17)$$

$$r\sin f = a\sin E\sqrt{1 - e^2} \quad (18)$$

Therefore, at $t = t^*$, when $f = \pi/2$,

$$r = a(1 - e^2) \quad (19)$$

Thus, from Eqs. (17) and (18), at time $t = t^*$,

$$\sin E = \sqrt{1 - e^2} \quad (20)$$

$$\cos E = e \quad (21)$$

and Eq. (15) becomes

$$nt^* = \cos^{-1}(e) - e\sqrt{1 - e^2} \quad (22)$$

Substituting Eq. (22) into Eq. (14) gives the following relationship between e and i:

$$\pi/2 - \cos^{-1}(e) + e\sqrt{1 - e^2} = i \quad (23)$$

or equivalently

$$e\sqrt{1 - e^2} + \sin^{-1}(e) = i \quad (24)$$

We claim that this condition, along with Eqs. (2) and (3), is sufficient to give a roughly circular ground track. A graph of Eq. (24) is shown in Fig. 4. The near linearity of this condition is easily seen from the expansion

$$i = \sin^{-1}(e) + e\sqrt{1 - e^2}$$

$$= \left(e + \frac{e^3}{6} + \frac{3e^5}{40} + \cdots\right) + e\left(1 - \frac{e^2}{2} - \frac{e^4}{8} + \cdots\right)$$

$$i = 2e - \frac{e^3}{3} - \frac{e^5}{20} + \cdots$$

and for small eccentricities, $i \doteq 2e$, where i is in radians.

Derivation of Ground Track Equation

We now derive an expression for the ground track of a geosynchronous (24 h period) satellite. Actually, this means that the period $P = 23$ h, 56 min, 04 s, which is the sidereal period of the axial rotation of the Earth. From Eq. (16) and the fact that

$$r = a(\cos E - e)(\cos f)^{-1}$$

we get

$$\cos E = (e + \cos f)(1 + e\cos f)^{-1} \quad (25)$$

and

$$\sin E = \sqrt{1 - e^2}\sin f(1 + e\cos f)^{-1} \quad (26)$$

With θ_G, θ_r, t, n, and θ as before, Eqs. (10) and (15) yield

$$\theta_r = -\theta_G - E + e\sin E$$

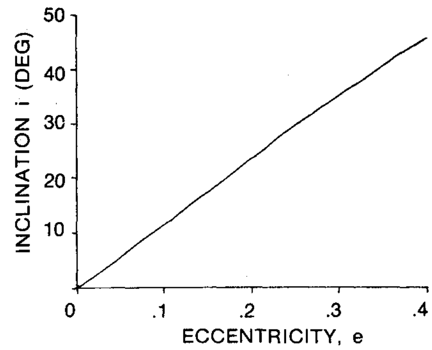

Fig. 4 Circular ground track condition.

and using Eqs. (25) and (26),

$$\theta_r = \theta - \theta_G - \cos^{-1}\left(\frac{e + \cos f}{1 + e\cos f}\right) + \frac{e\sqrt{1 - e^2}\sin f}{1 + e\cos f} \quad (27)$$

From Eq. (5) with $\Omega = 0$ and Eq. (6),

$$\cos\omega\cos f - \sin\omega\sin f = \cos\phi\cos\theta$$

and

$$\sin\omega\cos f + \cos\omega\sin f = \sin\phi/\sin i$$

from which it follows that

$$\sin f = \sin\phi\cos\omega/\sin i - \cos\phi\cos\theta\sin\omega \quad (28)$$

and

$$\cos f = \cos\phi\cos\theta\cos\omega + \sin\phi\sin\omega/\sin i \quad (29)$$

Using Eqs. (28) and (29), Eq. (27) becomes

$$\theta_r = \theta - \theta_G - \cos^{-1}\left(\frac{e + \cos\phi\cos\theta\cos\omega + \dfrac{\sin\phi\sin\omega}{\sin i}}{1 + e\cos\phi\cos\theta\cos\omega + \dfrac{e\sin\phi\sin\omega}{\sin i}}\right)$$

$$+ \frac{e\sqrt{1 - e^2}\left(\dfrac{\sin\phi\cos\omega}{\sin i} - \cos\phi\cos\theta\sin\omega\right)}{1 + e\cos\phi\cos\theta\cos\omega + \dfrac{e\sin\phi\sin\omega}{\sin i}} \quad (30)$$

where, from Eq. (7),

$$\theta = \sin^{-1}(\tan\phi/\tan i) \quad (31)$$

If we let $\omega = \pi/2$ to assure symmetry, Eq. (30) reduces to

$$\theta_r = \sin^{-1}\left(\frac{\tan\phi}{\tan i}\right) - \theta_G - \cos^{-1}\left(\frac{e + \dfrac{\sin\phi}{\sin i}}{1 + \dfrac{e\sin\phi}{\sin i}}\right)$$

$$- \frac{e\sqrt{1 - e^2}\cos\phi\left(1 - \dfrac{\tan^2\phi}{\tan^2 i}\right)^{\frac{1}{2}}}{1 + \dfrac{e\sin\phi}{\sin i}} \quad (32)$$

This gives the longitude coordinate θ_r for the latitude coordinate ϕ. A graph of the ground track for $i = \pi/6$ and $e = 0.265$ is in Fig. 2.

Error Analysis

We now consider the question of how close to a circle is the ground track as given by Eq. (32).

Define $\Delta(\phi)$ as the difference between the actual ground track longitude and the longitude to the circle for a given latitude ϕ. That is,

$$\Delta(\phi) = \theta_r - \theta'$$

where θ' is the longitude of the circle for a given ϕ, so $\theta' = \cos^{-1}(\cos i/\cos\phi)$. Then,

$$\Delta(\phi) = \cos^{-1}\left(\frac{\tan\phi}{\tan i}\right) - \cos^{-1}\left(\frac{e + \frac{\sin\phi}{\sin i}}{1 + \frac{e\sin\phi}{\sin i}}\right)$$

$$- \frac{e\sqrt{1-e^2}\cos\phi\left(1 - \frac{\tan^2\phi}{\tan^2 i}\right)^{\frac{1}{2}}}{1 + \frac{e\sin\phi}{\sin i}} - \cos^{-1}\left(\frac{\cos i}{\cos\phi}\right) \tag{33}$$

We note that $\Delta(\phi) = 0$ at $\phi = -i$, 0, or i, indicating that the ground track is exactly on the circle at those points. This is clear since that is how the ground track was developed [see Eq. (14)]. To consider the "size" of $\Delta(\phi)$ for other values of ϕ, we use three measures. The first two are the root mean square error (RMSE) and the mean absolute error (MAE). Thus,

$$\text{RMSE} = \left[\frac{1}{2N}\sum\Delta(\phi)^2\right]^{\frac{1}{2}} \tag{34}$$

and

$$\text{MAE} = \frac{1}{2N}\sum|\Delta(\phi)| \tag{35}$$

where the sums are from $\phi = -i$ to i in steps of i/N.

In order to "normalize" the measures of error, we divide by i and so also consider

$$\frac{\text{RMSE}}{i} = \frac{1}{i}\left[\frac{1}{2N}\sum\Delta(\phi)^2\right]^{\frac{1}{2}} \tag{36}$$

and

$$\frac{\text{MAE}}{i} = \frac{1}{2Ni}\sum|\Delta(\phi)| \tag{37}$$

with i in radians. The results are shown in Table 1 ($N = 100$).

These calculations show remarkably good agreement of the ground track to the circle, particularly for small i. In fact, it can be shown analytically that

$$\lim\Delta(\phi) = 0, \qquad i \to 0$$

and

$$\lim(\theta_r/\theta') = 1, \qquad i \to 0$$

Hence, the root mean square error and the mean absolute error approach zero as i tends to zero.

Consider now a third measure of the error, the area $A(i)$ between the ground track and the circle. Then, using Eq. (33),

$$A(i) = \int_{-i}^{i}|\Delta(\phi, i)|\,d\phi \tag{38}$$

so that

$$A(i) = \int_0^i[\Delta(\phi, i) - \Delta(-\phi, i)]\,d\phi$$

Table 1 Normalized measures of error

i, deg	i, rad	e	RMSE	MAE	RMSE/i	MAE/i
0	0	0	0	0	0	0
2.5	0.043633	0.021818	0.000089	0.000082	0.002032	0.001878
5	0.087266	0.043647	0.000353	0.000323	0.004040	0.003704
7.5	0.130900	0.065477	0.000799	0.000724	0.006100	0.005532
10	0.174533	0.087378	0.001422	0.001287	0.008150	0.007376
15	0.261799	0.131278	0.003257	0.002893	0.012441	0.011052
20	0.349066	0.175437	0.005969	0.005196	0.017101	0.014885
25	0.436332	0.219953	0.009690	0.008225	0.022208	0.018851
30	0.523599	0.264932	0.014600	0.012039	0.027885	0.022992

Hence, the normalized area is

$$\frac{A(i)}{i^2} = \int_0^i \frac{\Delta(\phi, i) - \Delta(-\phi, i)}{i^2}\,d\phi \tag{39}$$

Now,

$$\Delta(\phi, i) - \Delta(-\phi, i) = \cos^{-1}\left(\frac{\tan\phi}{\tan i}\right) - \cos^{-1}\left(\frac{-\tan\phi}{\tan i}\right)$$

$$-\cos^{-1}\left(\frac{e + \frac{\sin\phi}{\sin i}}{1 + \frac{e\sin\phi}{\sin i}}\right) + \cos^{-1}\left(\frac{e - \frac{\sin\phi}{\sin i}}{1 - \frac{e\sin\phi}{\sin i}}\right)$$

But using Eq. (39) and the Mean Value Theorem, there exists some ϕ_0, $0 \le \phi_0 \le i$ so that

$$\frac{A(i)}{i^2} = \frac{\Delta(\phi_0, i) - \Delta(-\phi_0, i)}{i}$$

and

$$\lim_{i\to 0}\frac{A(i)}{i^2} = \lim_{i\to 0}\frac{\Delta(\phi_0, i) - \Delta(-\phi_0, i)}{i}$$

Let $\phi_0 = \lambda i$ and $0 \le \lambda \le 1$. We assume that $\lambda = \lambda(i)$ is a differentiable function. Then, for $|i| \ll 1$,

$$\lim_{i\to 0}\frac{A(i)}{i^2} = \lim_{i\to 0}\frac{1}{i}\left[\cos^{-1}(\lambda) - \cos^{-1}(-\lambda) - \cos^{-1}\left(\frac{e+\lambda}{1+e\lambda}\right) + \cos^{-1}\left(\frac{e-\lambda}{1-e\lambda}\right)\right]$$

Note that since $i = \sin^{-1}(e) + e\sqrt{1-e^2}$, for $|i| \ll 1$, $i \doteq 2e$. Hence,

$$\lim_{i\to 0}\frac{A(i)}{i^2} = \lim_{e\to 0}\frac{A(2e)}{4e^2}$$

This limit is indeterminate (0/0), but a straightforward application of l'Hopital's rule yields the result

$$\lim_{i\to 0}\frac{A(i)}{i^2} = 0$$

Thus $A(i)$, the difference in area between the actual ground track and a true circle, approaches zero faster than i^2, as $i \to 0$. Similar analysis shows that the normalized root mean square error and the normalized mean absolute error also tend to zero; thus $\Delta(\phi)$ goes to zero faster than i as $i \to 0$.

Applications

The advantages of placing satellites in geosynchronous orbits are well known. U.S. communication and weather satellites

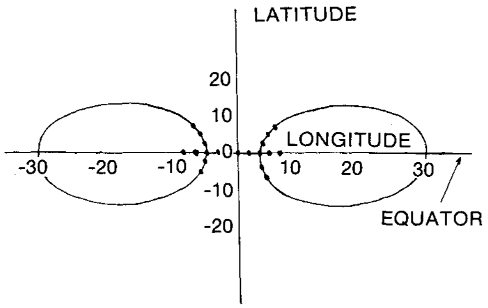

Fig. 5 Enhanced capacity.

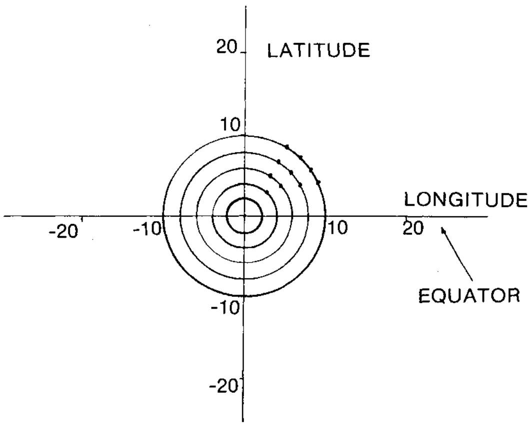

Fig. 6 Greatly enhanced capacity.

are usually placed in geosynchronous, circular, equatorial orbits over (or nearly over) South America. From this position in space, they are always in view of any ground station in the United States. However, there is an angular limit as to how closely satellites can be placed in such an orbit and not interfere with each other's transmissions to and from ground stations. This limit is about 1 deg. According to Van der Ha,[6] the population of geosynchronous satellites is growing at an annual rate of approximately 20/year.

Several approaches have been made to alleviate this problem and allow more satellites to be used. Rowe and Penzias[7] proposed using satellites with ground tracks in a figure "eight" that employed a zero eccentricity in a geosynchronous orbit. Because of the difficulty in maintaining the proper phase relationships, Shinji and Kurose[8] proposed using satellites with simple (no cross points) closed ground tracks. They suggested two methods. First, it was proposed to use satellites with ground tracks as shown below in Fig. 5 to increase the capacity roughly three times, (assuming a ± 30 deg range and a 1 deg separation of all satellites). The satellites are positioned along the equator (with the exception of the points where the ellipses cross) with 1 deg separation and around the two ellipses with 1 deg separation. Second, they suggested using a system of satellites with concentric circular ground tracks (see Fig. 6). In this system, the capacity was increased over eight times using a 2 deg separation in each orbit and a 2 deg separation of orbits. (Not all of the concentric orbits are shown in Fig. 6.) Here, the satellites are separated by 2 deg in each circle like beads on a ring.

Each of these ideas employ roughly circular ground tracks. The analytical method derived in this paper could be used in the preliminary design of these orbits, using the inclination to obtain the necessary eccentricity.

Because the tracks are not circular, the satellite is not north of the equator exactly half the time. For example, if the satellite orbit has its perigee at the southern extreme ($\omega = -\pi/2$), the satellite is in the northern hemisphere for time (in hours),

$$\frac{24}{\pi}\left[\frac{\pi}{2} + \sin^{-1}(e) + e\sqrt{1-e^2}\right]$$

for any geosynchronous satellite and in view of Eq. (24), the time (in hours) is

$$12 + 24(i/\pi)$$

for $0 \leq i \leq \pi/2$. Thus, a larger inclination gives more time to be observed, however, too large an inclination means the eccentricity is large and the satellite may be so far out that noise becomes a problem.

Conclusions

Sufficient conditions on the Keplerian elements to produce a near-circular ground track of a satellite in Earth orbit have been derived. The most significant of these conditions is an analytic expression relating inclination and eccentricity.

The application of this type of analysis to ground tracks of other shapes should be straightforward in many cases. An obvious example is pear-shaped orbits with relatively long stays above the equator. Another interesting avenue of future work in this area is a topological characterization of those types of curves actually achievable as ground tracks and those that are not.

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